General solution of overdamped Josephson junction equation in the case of phase-lock

S.I. Tertychniy
VNIIFTRI, Mendeleevo, Moscow Region, 141570, Russia

Abstract

The first order nonlinear ODE $\dot{\varphi}(t)+\sin\varphi(t)=B+A\cos\omega t$, ($A, B, \omega$ are real constants) which is commonly used as a simple model of an overdamped Josephson junction in superconductors is investigated. Its general solution is obtained in the case of the choice of parameters associated with one of three possible kinds of asymptotic behavior of solutions known as phase-lock where all but one solutions converge to a common 'essentially periodic' attractor. The general solution is represented in explicit form in terms of the Floquet solution of a particular instance of the double confluent Heun equation (DCHE). In turn, the solution of DCHE is represented through the Laurent series which defines an analytic function on the Riemann sphere with punctured poles. The Laurent series coefficients are given in explicit form in terms of infinite products of 2×2 matrices with a single zero element. The closed form of the phase-lock condition is obtained which is represented as the condition of existence of a real root of the transcendental function. The phase-lock criterion is conjectured whose plausibility is confirmed in numerical tests.

1 Introduction

The nonlinear first order ODE

$$\dot{\varphi}(t)+\sin\varphi(t)=q(t), \tag{1}$$

is commonly used in applied physics as the simple mathematical model describing the electric properties of Josephson junctions (JJ's) in superconductors [1, 2]. Here the RHS function $q(t)$ assumed to be known specifies the external impact to JJ representing the appropriately normalized *bias current* (or simply *bias*) supplied by an external current source. The unknown real valued function $\varphi(t)$ called *the phase* describes the macroscopic quantum

state of JJ. In particular, it is connected with the instantaneous voltage V applied across JJ in accord with the equation $V = (\hbar/2e)\mathrm{d}\varphi/\mathrm{d}\tau$, where $\hbar$ is the Plank constant, e is the electron charge, τ is the (dimensional) current time. The dimensionless variable t entering Eq. (1) is defined as $t = \omega_c\tau$, where ω_c is a constant parameter depending on the junction properties and named JJ characteristic frequency. See Ref [5] for more details of JJ physics.

Eq. (1) arises as the limiting case of the second order ODE utilized in more general Resistively Shunted Junction (RSJ) model [3, 4]. The reduction is legitimate if the role of the junction capacitance proves negligible. In practice, if JJ can be described by Eq. (1) it is named *overdamped*. Summarizing the aforementioned relationships, we shall name Eq. (1), for brevity, *overdamped Josephson junction equation* (OJJE).

Under concordant conditions, the theoretical modeling applying OJJE is in excellent agreement with experiments. It is also worth noting that nowadays electronic devices based on the Josephson effect play the important role in various branches of measurement technology. In particular, JJ arrays serve the heart element of the modern DC voltage standards [6]. The development of JJ-based synthesizers of AC voltage waveforms is currently in progress [7, 8, 9]. These and other successful applications stimulate the growing interest to the theoretical study and the modeling of JJ properties including investigation of capabilities of RSJ model and its limiting cases and the predictions they lead to.

To be more specific, the case most important from viewpoint of applications and simultaneously distinguished by the wealthiness of the underlaid mathematics is definitely the one of the bias function representing harmonic oscillations. Without loss of generality, it can be get in the following form

$$q(t) = B + A\cos\omega t, \tag{2}$$

where $A, B, \omega > 0$ are some real constant parameters. Hereinafter, the abbreviation OJJE introduced above will refer to the couple (1),(2).

In spite of apparent simplicity of OJJE, few facts of its specific analytic theory had been available until recently. Some preliminary results concerning the problem of derivation of analytic solutions of OJJE in general setting had been obtained in Ref. [12]. The approach put forward therein is elaborated in the present work. The focus is made on the case of manifestation of the *phase-lock* property by Eq. (1) which is one of its most important features from viewpoint of applications. The phase-lock property is formalized as follows:

in the case of phase-lock any solution $\varphi = \varphi(t)$ of Eq. (1) either yields a periodic exponent $e^{\mathrm{i}\varphi}$, exactly two such distinguished solutions existing, or, as the time parameter grows, $e^{\mathrm{i}\varphi}$ exponentially converges to the similar exponent for the one (common for all φ's) of the periodic functions just noted (another one is the repeller). (Here and in what follows we shall not distinguish phase functions which differ by a constant equal to 2π times an integer.) The corresponding period coincides with one of $q(t)$, i.e., in the case (2), $2\pi\omega^{-1}$. This behavior is stable with respect to weak parameter perturbations, i.e. the subset of parameter values leading to phase-lock is open. It is worth noting for completeness that in the opposite (no phase-lock) case no *stable* periodicity in the behavior of $e^{\mathrm{i}\varphi}$ is observed. There is also a third, intermediate type of the phase behavior, where the attractor and repeller are, in a sense, merged. It is realized on the lower-dimensional subset of the space of parameter values [11].

In the present work, the complete analytic solution of OJJE is obtained under assumption of the parameter choice ensuring the phase-lock property. The closed form of the phase-lock criterion in the form of the constraint imposed on the problem parameters is conjectured.

2 Overdamped Josephson junction equation against reduced double confluent Heun equation

The analytic theory of OJJE (1),(2) can be based on its reduction to the following system of two *linear* first order ODEs [12]

$$
\begin{aligned}
4\mathrm{i}\omega z^2\, x'(z) &= 2zx(z) + \left[2Bz + A\left(z^2+1\right)\right] y(z), \\
-4\mathrm{i}\omega z^2\, y'(z) &= \left[2Bz + A\left(z^2+1\right)\right] x(z) + 2zy(z),
\end{aligned} \tag{3}
$$

where z is the free complex variable. Indeed, on the universal covering $\Omega_1 \simeq \mathbb{R} \ni t$ of the unit circle in $\mathbb{C}$, i.e. for $z = \exp(\mathrm{i}\omega t)$, any non-trivial solution of Eqs. (3) determines a solution of OJJE in accordance with the equation

$$
\exp(\mathrm{i}\varphi) = \frac{\Re x - \mathrm{i}\Re y}{\Re x + \mathrm{i}\Re y} \tag{4}
$$

($\Re$ denotes the real part) supplemented with the continuity requirement. [The fulfillment of OJJE follows from a straightforward computation taking

into account the equality $\overline{z} = z^{-1}$ holding true on Ω_1 which is utilized for the demonstration that, for real A, B, ω, the functions $\Re x, \Re y$ also verify Eqs. (3) on Ω_1.] Conversely, any real valued solution $\varphi(t)$ of OJJE induces through Eq. (4) some 'initial data' $x(0), y(0)$ (with arbitrary norm $(x^2(0)+y^2(0))^{1/2} > 0$) for Eqs. (3). Having solved the latter on Ω_1 for x, y, one obtains, applying (4), another phase function obeying OJJE. It however must coincide, due to the identical initial value assumed at $t = 0$, with the original $\varphi(t)$. Finally, since these x, y defined on Ω_1 simultaneously obey the linear ODEs (3) with meromorphic coefficients which have the only singular points $z = 0$ and $z^{-1} = 0$, they can be extended, integrating (3) along radial directions, to analytic functions defined on the whole $\Omega = \Omega_1 \times \mathbb{R}_+$, the universal covering of the Riemann sphere with the punctured poles $z = 0$ and $z^{-1} = 0$.

It is worth noting that on Ω_1 the real valued functions $\tilde{x} = \tilde{x}(t) = \Re x(e^{\mathrm{i}\omega t}), \tilde{y} = \tilde{y}(t) = \Re y(e^{\mathrm{i}\omega t})$ satisfy the equations

$$2\,\mathrm{d}\,\tilde{x}/\mathrm{d}t = \tilde{x} + q\tilde{y},\ \ -2\,\mathrm{d}\,\tilde{y}/\mathrm{d}t = q\tilde{x} + \tilde{y}. \tag{5}$$

leading, together with (4), to the equation $(\mathrm{d}/\mathrm{d}t) log(\tilde{x}^2 + \tilde{y}^2) = \cos\varphi$ which implies

$$\mathrm{const}_1\, e^{-t} \leq |\tilde{x} + \mathrm{i}\tilde{y}|^2 \leq \mathrm{const}_2\, e^{t} \text{ for } t > 0. \tag{6}$$

Notice that if $x, y \not\equiv 0$ then $\mathrm{const}_1, \mathrm{const}_2$ may be assumed to be strictly positive. We shall refer to these boundings later on.

The key observation enabling one to radically simplify the problem of description of the space of solutions of Eqs. (3) is as follows [12]. Let us introduce the analytic function $v(z)$ which satisfies the equation

$$\left[z^2\frac{\mathrm{d}^2}{\mathrm{d}^2 z} + \big(\mu(z^2+1) - nz\big)\frac{\mathrm{d}}{\mathrm{d}z} + (2\omega)^{-2}\right] v = 0, \tag{7}$$

where

$$n \ = \ -\left(\frac{B}{\omega} + 1\right),\ \mu = \frac{A}{2\omega}, \tag{8}$$

are the constants replacing original A, B which will be used below whenever it proves convenient. Then a straightforward calculation shows that the functions x, y determined by the equations

$$v \ = \ \mathrm{i}\, z^{\frac{n+1}{2}} \exp\left(\frac{1}{2}\mu\left(-z + z^{-1}\right)\right)(x - \mathrm{i}y), \tag{9}$$

$$v' \ = \ (2\omega z)^{-1} z^{\frac{n+1}{2}} \exp\left(\frac{1}{2}\mu\left(-z + z^{-1}\right)\right)(x + \mathrm{i}y) \tag{10}$$

verify Eqs. (3). Conversely, defining the function $v(z)$ through the solution x, y of Eqs. (3) in accordance with Eq. (9), a straightforward computation proves satisfaction of Eq. (10) and, then, Eq. (7) follows. Thus, Eqs. (3) are equivalent to (7) and Eqs. (9),(10) represent the corresponding one-to-one transformation.

Eq. (7) coincides, after appropriate identification of the constant parameters, with Eq. (1.4.40) from Ref. [15]. It represents therefore a particular instance of the double confluent Heun equation (DCHE) which can be shown to be in our case non-degenerated.

It is also worth reproducing here the *canonical form* of the "generic" DCHE as it is given in Ref. [15] (Eq. (4.5.1)). It reads

$$z^2\frac{\mathrm{d}^2y}{\mathrm{d}z^2} + (-z^2 + cz + t)\frac{\mathrm{d}y}{\mathrm{d}z} + (-az + \lambda)y = 0 \tag{11}$$

where a, c, t, λ are some constants. To adjust it to our case, a single term has to be eliminated setting $a = 0$. For brevity, we shall name this subclass of DCHE's *reduced.* Besides, some obvious rescaling of the free variable z is to be carried out. After these, the three constant parameters remained in the resulting equation correspond to our constant parameters n, μ, ω (we shall not need and so omit the reproducing of the concrete form of this transformation).

The general analytic theory of DCHE is given in the chapter 8 of the treatise [14]. DCHE solutions are there represented, up to nonzero factors given in explicit form, through the Laurent series whose coefficients are assumed to be computable through the 'endless' chain of 3-term linear homogeneous equations ('recurrence relations'). In the present work, we derive the solution of reduced DCHE in a cognate but more explicit form.

As a technical limitation, we also stipulate in the present work for the additional condition to be imposed on the free constant problem parameters claiming of them the ensuring of the phase-lock property. On the base of practice of numerical computations, it can be conjectured that such parameter values fill up a non-empty open subset (*phase-lock area*) in the whole parameter space (see also the Conjecture A below). The case where the parameters belong to its complement is left beyond the scope of the present work.

3 Formal solution of reduced DCHE by Laurent series

Let us introduce yet another unknown function $E(z)$ replacing $v(z)$ by means of the transformation

$$v(z) = z^{\frac{n+\epsilon}{2}-\mathrm{i}\varkappa} e^{-\mu z} E(z), \tag{12}$$

where the discrete *parity parameter* ϵ may assume one of the two values, either $\epsilon = 0$ or $\epsilon = 1$, and $\varkappa$ is some *real* positive constant which will be determined latter on. For v obeying (7), $E(z)$ verifies the equation

$$\begin{aligned} 0 \;\; = \;\; & z^3 E'' + z\left[(\epsilon - 2\mathrm{i}\varkappa)\, z - \mu\left(z^2 - 1\right)\right] E' \\ & + \left[\mu\left(\frac{n-\epsilon}{2} + \mathrm{i}\varkappa\right) z^2 \right. \\ & \quad + \left((1-\epsilon)\left(\frac{1}{4} + \mathrm{i}\varkappa\right) - \varkappa^2 - \left(\frac{n+1}{2}\right)^2 + \lambda\right) z \\ & \quad \left. + \mu\left(\frac{n+\epsilon}{2} - \mathrm{i}\varkappa\right)\right] E, \end{aligned} \tag{13}$$

where $\lambda = (2\omega)^{-2} - \mu^2$.

Conversely, (13) implies the fulfillment of Eq. (7).

At first glance, Eq. (13) seems 'much worse' than the original DCHE representation. Nevertheless, it is this equation which we shall attempt to solve searching for its solution in the form of Laurent series

$$E = \sum_{k=-\infty}^{\infty} a_k z^k \tag{14}$$

'centered' in the points $z = 0$ and $z^{-1} = 0$ (which are the only singular points for Eq. (13)) with unknown z-independent coefficients a_k. Then, carrying out straightforward substitution, one gets a sequence of 3-term recurrence relations binding triplets of neighboring series coefficients which can be written

down either as

$$
\begin{aligned}
0 &= -\mu\left(k-1-\frac{n-\epsilon}{2}-\mathrm{i}\varkappa\right)a_{k-1} \\
&\quad +(Z_k+\lambda)a_k+\mu\left(k+1+\frac{n+\epsilon}{2}-\mathrm{i}\varkappa\right)a_{k+1}, \qquad (15)\\
&\quad \text{where } Z_k=\left(k+\frac{\epsilon-1}{2}-\mathrm{i}\varkappa\right)^2-\left(\frac{n+1}{2}\right)^2, \qquad (16)
\end{aligned}
$$

or as

$$
\begin{aligned}
0 &= -\mu\left(k-1-\frac{n+\epsilon}{2}+\mathrm{i}\varkappa\right)a_{-(k-1)} \\
&\quad +(\tilde{Z}_k+\lambda)a_{-k}+\mu\left(k+1+\frac{n-\epsilon}{2}+\mathrm{i}\varkappa\right)a_{-(k+1)}, \qquad (17)\\
&\quad \text{where } \tilde{Z}_k=\left(k+\frac{1-\epsilon}{2}+\mathrm{i}\varkappa\right)^2-\left(\frac{n+1}{2}\right)^2 \qquad (18)
\end{aligned}
$$

and $k=0,\pm1,\pm2\ldots$. The sets of Eqs. (15) and (17) are exactly equivalent and each of them covers the whole set of equations the coefficients a_k have to obey. However, in the approach utilized in the present work, we shall consider them both in conjunction, employing (15) for coefficients a_k with $k\geq-1$ and (17) for a_k with $k\leq1$. Thus, Eqs. (15) and (17) will be considered separately but on the common index variation 'half-interval' $k\geq0$ (remaining legitimate, in principle, for arbitrary integer k). Obviously, these two equation sets cover the complete set of conditions imposed to the coefficients a_k and are 'almost disjoined' intersecting in their 'boundary' $k=\pm0$-members alone.

Let us further consider for $k\geq0$ the following *formal infinite products*

$$R_k=\prod_{j=k}^{\infty}M_j,\ \ \tilde{R}_k=\prod_{j=k}^{\infty}\tilde{M}_j \qquad (19)$$

of the 2×2 matrices

$$
\begin{aligned}
M_j &= \begin{pmatrix} 1+\lambda/Z_j & \mu^2/Z_j \\ 1 & 0 \end{pmatrix}, \qquad (20)\\
\tilde{M}_j &= \begin{pmatrix} 1+\lambda/\tilde{Z}_j & \mu^2/\tilde{Z}_j \\ \tilde{Z}_{j-1}/\tilde{Z}_j & 0 \end{pmatrix}. \qquad (21)
\end{aligned}
$$

It is assumed throughout that the matrices $M_j, \tilde{M}_j$ with *larger* indices j are situated in products *to the right* with respect to ones labeled with *lesser* index values.

Notice that in the case $\varkappa = 0$ and integer n, zero may appear in denominators of M-factors, making the above definitions meaningless. This apparent fault admits a simple resolution (see Eq. (38) and the discussion following it). For a while, we temporary leave out consideration of such specific parameter choices.

It is also worth noting that the above definitions of R_k, $\tilde{R}_k$ may be understood as a concise form of representation of the 'descending' recurrence relations

$$R_k = M_k R_{k+1}, \tag{22}$$
$$\tilde{R}_k = \tilde{M}_k \tilde{R}_{k+1}, \; k = 0, 1, \ldots \tag{23}$$

among the neighboring R's. These are the only dependencies which will be actually used below in derivations involving $R_k, \tilde{R}_k$.

The formulas (19) are 'formal' since neither the issue of the convergence of such sequences nor how one has to understand the 'initial values' $R_\infty, \tilde{R}_\infty$ necessary for the actual determination of the 'finite index value' R-matrices are here addressed.

Now a straightforward calculation applying Eqs. (22),(23) shows that the following formulas

$$\tilde{a}_k = \mu^k \frac{\Gamma\left(1 + \frac{n+\epsilon}{2} - \mathrm{i}\varkappa\right)}{\Gamma\left(k + 1 + \frac{n+\epsilon}{2} - \mathrm{i}\varkappa\right)} (0, 1) \cdot R_k \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{24}$$

$$\tilde{a}_{-k} = \frac{\mu^k}{\tilde{Z}_{k-1}} \frac{\Gamma\left(1 + \frac{n-\epsilon}{2} + \mathrm{i}\varkappa\right)}{\Gamma\left(k + 1 + \frac{n-\epsilon}{2} + \mathrm{i}\varkappa\right)} (0, 1) \cdot \tilde{R}_k \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{25}$$

yield *the formal solutions* to Eqs. (15) and (17), respectively.

4 Validation of the formal solution

In this section we show that the formal solution of Eq. (13) presented in the form of expansion (14) with coefficients given by Eqs. (24),(25) represents a well defined analytic function of z. This means, first of all, that the infinite matrix products $R_k, \tilde{R}_k$ it involves converge. Moreover, the convergence takes place for any constant parameter values.

The key auxiliary result which may be utilized for the proof of this assertion is as follows.

Lemma.

Let us consider the sequences of complex numbers $\alpha_j, \beta_j, \gamma_j, \delta_j$ satisfying the following 'ascending' matrix recurrence relation

$$\begin{pmatrix} \alpha_j & \beta_j \\ \gamma_j & \delta_j \end{pmatrix} = \begin{pmatrix} \alpha_{j-1} & \beta_{j-1} \\ \gamma_{j-1} & \delta_{j-1} \end{pmatrix} \begin{pmatrix} 1+\lambda/Z_j & \mu^2/Z_j \\ \sigma_j & 0 \end{pmatrix}, \tag{26}$$

where

$$\begin{aligned} \text{either } \sigma_j &= 1 \\ \text{or } \sigma_j &= Z_{j-1}/Z_j. \end{aligned} \tag{27}$$

Then all they converge as $j \to \infty$. Moreover, $\lim \beta_j = 0 = \lim \delta_j$ whereas for α, γ-sequences there exist positive quantities N_α, N_γ and positive integer j_0, all depending at most on $n, \mu, \lambda, \varkappa, \epsilon$, such that for all $j > j_0$

$$\begin{aligned} |\alpha_j - lim_{j' \to \infty} \alpha_{j'}| &< N_\alpha \max_{j' > j_0}(|\alpha_{j'}|) j^{-1}, \\ |\gamma_j - lim_{j' \to \infty} \gamma_{j'}| &< N_\gamma \max_{j' > j_0}(|\gamma_{j'}|) j^{-1}, \end{aligned} \tag{28}$$

where the maxima are finite.

The outline of the Lemma proof can be found in the Appendix.

Remark: Formally, we need not include in the lemma stipulation the requirement that either $\varkappa \neq 0$ or n is non-integer (which would a priori evade possibility of contributions with zero Z_* in denominator) because with fixed constant parameters and sufficiently large j_0 no zero Z_* may appear.

Let us return to Eqs. (19) and consider the four double-indexed sets of complex numbers $\{\alpha, \beta, \gamma, \delta\}_j^{(j_0)}$ defined as follows:

$$\begin{pmatrix} \alpha_j^{(j_0)} & \beta_j^{(j_0)} \\ \gamma_j^{(j_0)} & \delta_j^{(j_0)} \end{pmatrix} = R_{j_0}^{(j)} = \prod_{k=j_0}^{j} M_k. \tag{29}$$

It is straightforward to verify that the sequences obtained by the picking out the elements with common value of the upper index j_0 obey the recurrence relations (26) for the upper choice in (27). Hence it follows from the Lemma

that all they converge. We denote the corresponding limits as $\alpha^{(j_0)}$ etc. We have therefore the consistent definition for the infinite matrix products (19)

$$R_{j_0} = \begin{pmatrix} \alpha^{(j_0)} & \beta^{(j_0)} \\ \gamma^{(j_0)} & \delta^{(j_0)} \end{pmatrix}. \tag{30}$$

Let us further note that, increasing j_0, the 'starting' sequence elements ($\alpha^{(j_0)}_{j_0}, \alpha^{(j_0)}_{j_0+1}$ for α etc) tend to the corresponding elements of the idempotent matrix

$$M_\infty = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix},\ M_\infty^2 = M_\infty, \tag{31}$$

the discrepancy decreasing as $O(j_0^{-2})$. On the other hand, in accordance with (28) the elements of $R^{(j)}_{j_0}$ differ from their j-limits constituting R_{j_0} by $O(j_0^{-1})$-order quantities. This means that R_{j_0} tends to M_∞ as j_0 goes to infinity with the difference going to zero as $O(j_0^{-1})$. In other words, we have the

Lemma corollary:

1. $R_{j_0} - M_\infty = O(j_0^{-1})$.

Besides, in view of the convergence,

2. The modules of the elements of all the matrices R_{j_0} are bounded from above in total

— provided 'no-zeroes-in denominators' condition for the parameter choice is respected, of course.

In according with the above properties, decomposing R_j into the product of the two factors, $R_j = R_j^{j_0} \cdot R_{j_0}$, and approximating R_{j_0} by M_∞, one obtains a simple but important algorithm of computation of the products (19). The approximation

$$R_j \approx \prod_{k=j}^{j_0} M_k \cdot M_\infty, \tag{32}$$

is the better, the larger $j_0 > j$ is selected. In the limit, one gets

$$R_j = \begin{pmatrix} \alpha^{(j)} & 0 \\ \gamma^{(j)} & 0 \end{pmatrix}. \tag{33}$$

This interpretation resolves (in a quite obvious way, though) the aforementioned problem of specification of the 'initial value' for the sequence R_j treated through the 'descending' recurrence relation '$[R_\infty] \cdots \to R_j \to R_{j-1} \to \ldots$' implied by Eq. (22). (As a matter of fact, an arbitrary constant matrix whose product with M_∞ is nonzero, including the unit matrix, might be used instead of M_∞ in (32), affecting only the overall normalization of the result and the accuracy of (32)-like approximation for given j_0.)

Having introduced the consistent representation of the matrices (19), it is straightforward to do the same for the matrices $\tilde{R}_j$ (20). The above speculation applies to them with minor modifications as well. The only distinction is the making use of the second (lower) choice in Eq. (27) and the operating with complex conjugated quantities (equivalent in our case to the replacing $\varkappa$ by $-\varkappa$) throughout. We shall mark the elements of $\tilde{R}_j$ obtained in this way with *tildes* over the corresponding α's and γ's.

Now, with the consistent interpretation of the products $R_j, \tilde{R}_j$ (19) in hand, one is able to calculate the coefficients a_k, a_{-k} for $k = 0, 1, 2\ldots$ in accordance with Eqs. (24) (25). The triple matrix products reduce to separate elements of R's (or $\tilde{R}$'s) denoted above as $\gamma^{(k)}$ (for Eq. (24)), and $\tilde{\gamma}^{(k)}$(for Eq. (25), respectively) which are the functions of the parameters $n, \mu, \lambda, \varkappa, \epsilon$. Therefore, the sequences

$$\tilde{a}_k = \mu^k \gamma^{(k)} \frac{\Gamma\left(1 + \frac{n+\epsilon}{2} - \mathrm{i}\varkappa\right)}{\Gamma\left(k + 1 + \frac{n+\epsilon}{2} - \mathrm{i}\varkappa\right)} \tag{34}$$

$$\tilde{a}_{-k} = \frac{\mu^k \tilde{\gamma}^{(k)}}{\tilde{Z}_{k-1}} \frac{\Gamma\left(1 + \frac{n-\epsilon}{2} + \mathrm{i}\varkappa\right)}{\Gamma\left(k + 1 + \frac{n-\epsilon}{2} + \mathrm{i}\varkappa\right)}, \; k = 0, 1, 2, \ldots \tag{35}$$

are well defined and solve Eqs. (15),(17), respectively, for $k = 1, 2, \ldots$

The important feature of the expressions (34),(35) which is used below is their asymptotic behavior for large values of the index k which is easy to derive in explicit form. Specifically, in accordance with inequalities (28), the set of modules of γ- and $\tilde{\gamma}$-factors involved in Eqs. (34),(35) is bounded in total from above and each of these sequences converge to a finite limit. These imply in particular the validity of the estimates

$$|\tilde{a}_k| \propto \frac{\zeta_1^k}{k!}, \; |\tilde{a}_{-k}| \propto \frac{\zeta_2^k}{k^2 k!}, \tag{36}$$

asymptotically, in the leading order, for some k-independent ζ_1, ζ_2.

5 Matching condition and phase-lock criterion

By construction, the $\tilde{a}$-coefficients defined by Eqs. (24) and (25) obey the linear homogeneous equations (15) and (17), respectively, which are 'the same', essentially, differing only in the associated intervals of the variation of the index, consisting of the positive integers for the former and negative ones for the latter. However, these two sequences cannot be joined, automatically, since they are 'differently normalized'. This means, in particular, that their 'edge elements' indexed with zeroes, generally speaking, differ. We will denote them as $\tilde{a}_0$ and $\tilde{a}_{-0}$, respectively, distinguishing here, in notations, the index '0' from the index '-0'.

Now, referring to Eqs. (34),(35), one notes that in view of the factors of Γ-functions present in denominators and leading to asymptotic behaviors (36) the following power series in z

$$E_+(z) = 1 + \sum_{k=1}^{\infty} \frac{\tilde{a}_k}{\tilde{a}_0} z^k, \; E_-(z) = 1 + \sum_{k=1}^{\infty} \frac{\tilde{a}_{-k}}{\tilde{a}_{-0}} z^{-k}, \tag{37}$$

admit absolutely converging majorants. (We assumed above $\tilde{a}_{\pm 0} \neq 0$. Otherwise, i.e. if $\tilde{a}_{\pm 0} = 0$, $\tilde{a}_{\pm 1}$ may not vanish and the series with the coefficients $\tilde{a}_{\pm k}/\tilde{a}_{\pm 1}$ can be utilized instead.) Indeed, the Maclaurin series for the exponent can play this role. Therefore, the series $E_+(z)$ and $E_-(z^{-1})$ define some *entire functions* of z. As a consequence, the expression

$$E(z) = \frac{\frac{4}{\pi^2} \sin\left(\frac{\pi}{2}(n+\epsilon-2\mathrm{i}\varkappa)\right) \sin\left(\frac{\pi}{2}(n+\epsilon+2\mathrm{i}\varkappa)\right)}{(n+\epsilon-2\mathrm{i}\varkappa)(n+2-\epsilon+2\mathrm{i}\varkappa)} (E_+(z) + E_-(z) - 1) \tag{38}$$

represents a single-valued function analytic everywhere on the Riemann sphere except the poles $z = 0, z^{-1} = 0$. They are the essential singular points for E.

It has to be noted that the additional z-independent fractional factor in (38) may be regarded as a specific common 'normalization' of the (37)-type series which may be, in principle, arbitrary. However, its given form is, essentially, unique as being fixed (up to an insignificant nonzero numerical factor) in view of the following reasons.

The two sine-factors in the numerator regarded as holomorphic functions of $n + \epsilon \pm 2\mathrm{i}\varkappa$ are introduced for the canceling out zeroes in denominators arising due to the poles in the factors Z_j^{-1} and $\tilde{Z}_j^{-1}$ involved in the products

(22) and regarded as the functions of the same parameters. The set of these (vicious, essentially) singularities constitute a homogeneous grid which is just covered by the grid of roots of the sine-factors in (38) — with the two exceptions. These two 'superfluous' sine-factor roots are, in turn, 'neutralized' by the two linear factors in the denominator in (38) which are therefore also uniquely determined. As the result, in vicinity of any zero in denominators in coefficients of the power series defining $E(z)$ considered as the function of $n + \epsilon \pm 2\mathrm{i}\varkappa$ (a root of some Z_* or $\tilde{Z}_*$), the resulting expression takes the form of the ratio $\sin x/x$ ($x \simeq 0$) and is not now associated with any irregular behavior. Thus, as a matter of fact, the fractional factor involved in (38) is distinguished (up to a numerical factor) by the claims (i) to cancel out the poles in the original expressions of the $\tilde{a}$-coefficients (34), (35) considered as the functions of $n+\epsilon\pm 2\mathrm{i}\varkappa$ and (ii) to introduce neither more roots nor more poles as a result of such a 'renormalization'.

Now, when plugging the function (38) in Eq. (13) in order to verify its fulfillment, we may provisionally drop out, sparing the space, z-independent fractional factor (restoring it afterwards).

It is important to emphasize that the expressions (34), (35), by construction, verify all the 3-term recurrence relations (15),(17) which bind the a-coefficients with indices *of a common sign*, either non-negative or non-positive. The only equation which does not fall into the above categories, and, accordingly, has not been automatically fulfilled, is the 'central' one binding the coefficients $a_{-1}, a_0 = 1 = a_{-0}, a_1$, i.e. the equation

$$\mu\left(1+\frac{n-\epsilon}{2}+\mathrm{i}\varkappa\right)a_{-1}+(Z_0+\lambda)a_0+\mu\left(1+\frac{n+\epsilon}{2}-\mathrm{i}\varkappa\right)a_1=0. \quad (39)$$

With normalization adopted in (37), one has $a_0 = 1, a_1 = \tilde{a}_1/\tilde{a}_0, a_{-1} = \tilde{a}_{-1}/\tilde{a}_{-0}$. Further, in accordance with (34),(35)

$$\begin{aligned}\tilde{a}_0 &= \gamma^{(0)}, \tilde{a}_1 = \frac{\mu}{1+\frac{n+\epsilon}{2}-\mathrm{i}\varkappa}\gamma^{(1)}\\ \tilde{a}_{-0} &= \frac{\tilde{\gamma}^{(0)}}{\tilde{Z}_{-1}}, \tilde{a}_{-1} = \frac{\mu}{1+\frac{n-\epsilon}{2}+\mathrm{i}\varkappa}\frac{\tilde{\gamma}^{(1)}}{\tilde{Z}_0}\end{aligned}$$

Besides, one has $\gamma^{(0)} = \alpha^{(1)}$, $\tilde{\gamma}^{(0)} = \tilde{\alpha}^{(1)}\tilde{Z}_{-1}/\tilde{Z}_0$. Combining these dependencies, the following representation of Eq. (39) arises

$$0 = \mu^2\frac{\gamma^{(1)}}{\alpha^{(1)}}+(Z_0+\lambda)+\mu^2\frac{\tilde{\gamma}^{(1)}}{\tilde{\alpha}^{(1)}}. \quad (40)$$

Accordingly, it is convenient to introduce the following function of the parameters $\varkappa$ and $n, \lambda, \mu, \epsilon$

$$\begin{aligned}\Xi \;&=\; \frac{\frac{4}{\pi^2}\sin\left(\frac{\pi}{2}(n+\epsilon-2\mathrm{i}\varkappa)\right)\sin\left(\frac{\pi}{2}(n+\epsilon+2\mathrm{i}\varkappa)\right)}{(n+\epsilon-2\mathrm{i}\varkappa)(n+2-\epsilon+2\mathrm{i}\varkappa)}\times \\ &\quad \left(\mu^2\gamma^{(1)}\tilde{\alpha}^{(1)}+(Z_0+\lambda)\tilde{\alpha}^{(1)}\alpha^{(1)}+\mu^2\tilde{\gamma}^{(1)}\alpha^{(1)}\right) \end{aligned} \tag{41}$$

where $Z_0 = ((\epsilon-1)/2-\mathrm{i}\varkappa)^2 - ((n+1)/2)^2$ (see (16)) and α's, γ's are defined as the elements of the convergent matrix products as follows:

$$\begin{pmatrix} \alpha^{(1)} & 0 \\ \gamma^{(1)} & 0 \end{pmatrix} = \prod_{j=1}^{\infty} M_j, \; \begin{pmatrix} \tilde{\alpha}^{(1)} & 0 \\ \tilde{\gamma}^{(1)} & 0 \end{pmatrix} = \prod_{j=1}^{\infty} \tilde{M}_j \tag{42}$$

(see Eqs. (19)-(21)). The fractional multiplier in the first line of Eq. (41) coincides with the one entering Eq. (38) and plays the identical role: it eliminates the vicious singularities arising for specific values of the parameters $n, \varkappa$. We shall name $\Xi = \Xi(\varkappa, n, \mu, \lambda, \varkappa, \epsilon) \equiv \Xi(\varkappa; \dots)$ the *discriminant function* for brevity. The following statement holds true.

Proposition

> Restricting $\varkappa$ to real values, the equality $\Xi(\varkappa; \dots) = 0$ is the necessary and sufficient condition for the single valued analytic function (38) to verify Eq. (13) everywhere on the Riemann sphere except its poles $z = 0, z^{-1} = 0$.

Indeed, the vanishing of Ξ implies the fulfillment of (39) (where a's are expressed through $\tilde{a}$'s), the last equation binding coefficients of the expansion (14) which has not been fulfilled as the result of the very coefficients definition. Now all the 3-term recurrence relations for a-coefficients, which Eq. (13) is equivalent to, are satisfied and the analytic function (38) verifies Eq. (13) everywhere except of its own singular points $z = 0$ and $z^{-1} = 0$.

The equation

$$\Xi(\varkappa; \dots) = 0 \tag{43}$$

referred to in the above proposition can be named *the matching condition* since it enforces the sequences of the coefficients a_k, a_{-k}, separately obeying the corresponding 'halves' of the equation chain (15) (equivalently, (17)) to be 'matched' in their 'edge' elements $a_{\pm 0} = 1, a_{\pm 1}$.

It is worth emphasizing that, up to this point, the parameter $\varkappa$ (absent in Eq. (7)) has not been restricted in any way (it was only assumed to be real). Now and in what follows we regard the condition (43) as $\varkappa$ definition eliminating this odd 'degree of freedom'. Now it is a well defined function of the other parameters. It seems interesting enough that the addition of unspecified constant $\varkappa$ to the transformation (12) and its subsequent 'fine tuning' by means of the claim of fulfillment of Eq. (43) is necessary for the representation of solution of DCHE (7) in terms of convergent Laurent series. More precisely, it is clear that Eqs. (16) can be solved *for any* (including trivial zero) choice of $\varkappa$, choosing loosely a_{j_0}, a_{j_0+1} for arbitrary j_0 and then calculating, term by term, all the coefficients a_j, advancing in parallel in both directions of j-index variation 'from j_0 towards $\pm\infty$'. Then (14) immediately yields a ($\varkappa$-dependent!) formal solution to Eq. (13) and hence, through transformation (12), to Eq. (7). However, it can be only formal and will necessarily *diverge* for any z unless the matching condition (43) is fulfilled — just in view of the uniqueness of solution with the analytic properties presupposed. On the other hand, considering separately the 'halves' of the set of Eqs. (16) and resolving them 'in index variation directions' opposite to the ones assumed above (in a sense, 'from $\pm\infty$ towards ± 0'), we obtain the always *converging* series (37). However, as we have seen, we again have no solution (in this case even formal) unless the matching condition fixing $\varkappa$ is fulfilled. Obviously, the uniqueness property implies that the introduction of the 'branching' power function factor, as in (12), is the only way to obtain a solution to Eq. (7) admitting representation in terms of convergent power series.

Now, tracking back the relationship connecting Eq. (13) with the primary Eq. (1) and invoking the general theory of the latter applicable in the case of arbitrary continuous periodic $q(t)$ [11], one can infer the following statements which however, in the present context, are only of the status of *conjectures* in view of the lack of their proof 'from the first principles', i.e. on the base of the properties of the discriminant function Ξ following from its definition.

Conjecture A.

1. There exists an open non-empty subset of the space of the problem parameters n, μ, ω, ϵ where Eq. (43) admits a real valued positive solution.
2. If real $\varkappa$ solves Eq. (43), $-\varkappa$ is the solution as well. No more

real roots exist.

3. Real roots of Eq. (43) obey the condition $|\varkappa| \leq (2\omega)^{-1}$.

Remark: The last statement is nothing else but the form of the limitation on the rate of growth or decreasing of the functions $\tilde{x}, \tilde{y} = x, y|_{z=e^{\mathrm{i}\omega t}}$ of the real variable t implied by the inequalities (6) and the equation

$$x - \mathrm{i}y = -iz^{\frac{\epsilon-1}{2}-\mathrm{i}\varkappa} e^{-\frac{\mu}{2}\left(z+\frac{1}{z}\right)} E,$$

following from definitions. Notice that the latter clarifies the role of the *parity* parameter ϵ which determines the multiplicity of the inverse to the map $\mathbb{S}^1 \ni z \to (x + \mathrm{i}y)/|x + \mathrm{i}y| \in \mathbb{S}^1$ induced by the solution (38). If $\epsilon = 0$, the revolution along the circle $|z| = 1$ leads to the reversing of the direction of the vector with components (x, y) whereas in the case $\epsilon = 1$ its direction is preserved. The inverse map is double-valued in the former case and one-to-one in the latter one.

More properties of the discriminant function can be inferred from the numerical experiments although, as opposed to the assertions of the Conjecture A, they have, to date, no analytic arguments in their favor yet, even indirect. Nevertheless, the first item below is important enough from viewpoint of applications (seeming also plausible enough) to be *explicitly formulated* here.

Conjecture B.

1. **Phase-lock criterion.**

Equation (43) admits a real non-zero solution if and only if

$$\Xi(0; \dots) > 0. \tag{44}$$

This means in particular that $\Xi(0; \dots)$ is real; moreover, the numerical study makes evidence that

2. $\Xi(\varkappa; \dots)$ is real for real $\varkappa$

(assuming the other parameters to be also real, of course).

6 Floquet solutions of DCHE and involutive solution maps

Let us assume now that there exists a real positive solution $\varkappa$ of Eq. (43). With this $\varkappa$, the function $E(z)$ defined by Eq. (38) verifies Eq. (13). Let us

consider the function $E_{\#}(z)$ defined through $E(z)$ as follows:

$$E_{\#}(z) \quad = \quad z^{2i\varkappa-\epsilon}\left[E'\left(\frac{1}{z}\right)+\left(\left(\frac{n+\epsilon}{2}-i\varkappa\right)z-\mu\right)E\left(\frac{1}{z}\right)\right]. \quad (45)$$

Then a straightforward computation shows that it verifies Eq. (13), provided $E(z)$ does.

It is worthwhile to note that the repetition of the transformation (45) yields no more solutions to Eq. (13). As a matter of fact, one has $_{\#}\circ_{\#} = (2\omega)^{-2}Id$. Thus $(2\omega)^{-1}{}_{\#}$ is the involutive map on the space of its solutions.

Next, the functions $E(z)$ and $E_{\#}(z)$ are linearly independent for nonzero $\varkappa$. Indeed, utilizing (13), one finds

$$\begin{aligned} E_{\#}{}'(z) \quad = \quad & z^{2i\varkappa-\epsilon-1}\left[\left(-\frac{n+\epsilon}{2}+i\varkappa+\mu z\right)E'\left(\frac{1}{z}\right)+\right. \\ & \left.+\left(\mu\left(\frac{n+\epsilon}{2}-i\varkappa\right)(z^2+1)+\left(-\left(\frac{n+\epsilon}{2}-i\varkappa\right)^2+\lambda\right)z\right)E\left(\frac{1}{z}\right)\right]. \quad (46) \end{aligned}$$

This reduction allows one to calculate the determinant of the linear transformation binding the pairs of functions $E_{\#}, E_{\#}{}'(z)$ and $E, E'(1/z)$ which proves equal to $(2\omega)^{-2}z^{2(-\epsilon+2i\varkappa)}$ and is therefore nonzero. Hence $E_{\#}(z)$ is not identically zero (and may vanish at isolated points, at most, as well as $E(z)$). Finally, $E(z)$ is periodic on the unit circle centered at zero whereas $E_{\#}(z)$, for real $\varkappa \neq 0$, is not. Hence they are linear independent. The functions $E(z)$ and $E_{\#}(z)$ constitute therefore the fundamental system for Eq. (13) and any its solution can be expanded in this basis with constant expansion coefficients. The analytic properties of these solutions identify them as the unique pair of the *Floquet solutions* of the reduced DCHE under consideration. See [14], section 2.4.

The function $E(z)$ obeys the important functional equation which can be derived as follows. A straightforward calculation shows that the RHS expression of Eq. (45) with the 'branched' factor $z^{2i\varkappa}$ removed (i.e. $z^{-2i\varkappa}E_{\#}(z)$) satisfies the ODE which coincides with (13) *up to the opposite sign of the parameter* $\varkappa$. This means that, for real $\varkappa$, n, μ, ω, the analytic function

$$\hat{E}(z) \quad = \quad z^{-\epsilon}\left[\bar{E}'\left(\frac{1}{z}\right)+\left(\left(\frac{n+\epsilon}{2}+i\varkappa\right)z-\mu\right)\bar{E}\left(\frac{1}{z}\right)\right], \quad (47)$$

where $\bar{E}(z) = \overline{E(\overline{z})}$, is the solution of Eq. (13) itself.

As opposed to $E_\#$, this 'yet another' solution $\hat{E}(z)$ has *the same* analytic properties as $E(z)$ and hence must coincide with it up to some numerical factor C_C, i.e.

$$C_C E(z) = z^{-\epsilon}\left[\bar{E}'\left(\frac{1}{z}\right) + \left(\left(\frac{n+\epsilon}{2} + \mathrm{i}\varkappa\right) z - \mu\right) \bar{E}\left(\frac{1}{z}\right)\right]. \tag{48}$$

(C_C may not vanish since otherwise $E_\#$ would also be zero.) This is the generalization of the similar property of the so called 'Heun polynomials' established in Ref. [12].

The complex valued constant C_C actually reduces to a single real constant. To show that, let us notice at first that if $E(z)$ verifies Eq. (13) it follows from the latter and (48)

$$\begin{aligned} C_C E'(z) \;\; = \;\; & z^{-\epsilon-1}\Bigg[\left(-\frac{n+\epsilon}{2} - \mathrm{i}\varkappa + \mu z\right)\bar{E}'(1/z) \\ & + \Bigg(\mu\frac{\epsilon+n}{2}(z^2+1) - \mathrm{i}\mu\varkappa(z^2-1) \\ & + \left(-\frac{(n+\epsilon)^2}{4} - \varkappa^2 + \lambda\right) z\Bigg)\bar{E}(1/z)\Bigg]. \end{aligned} \tag{49}$$

Evaluating now Eqs. (48),(49) together with their complex conjugated versions with $z = z^{-1} = 1$, one obtains four linear homogeneous equations binding the quantities $E(1), E'(1), \bar{E}(1), \bar{E}'(1)$ which may not vanish simultaneously. The corresponding consistency condition reads $|C_C|^2 = (2\omega)^{-2}$ implying

$$C_C = (2\omega)^{-1} e^{\mathrm{i}C_c}, \tag{50}$$

where C_c is some *real* constant (actually, the function of the parameters n, μ, ω, ϵ). It encodes all the monodromy data for Eq. (13), essentially.

It is straightforward to show that the transformation Eq. (48) is also involutive. It manifests the specific symmetry in the behaviors of the function $E(z)$ in vicinities of the essentially singular points $z = 0$ and $z^{-1} = 0$. Remarkably, this symmetry implies itself the fulfillment of Eq. (13). Indeed, differentiating (48) and taking into account (50), one arrives at Eq. (13). In a sense, Eq. (48) together with stipulation for the analyticity of $E(z)$ can be considered as the equivalent to Eq. (13). Additionally, Eq. (48) implies anti-linear (involving complex conjugation) dependencies among the 'distant' Laurent series coefficients a_{-k} and a_k, a_{k+1} (whereas Eq. (15) (or (17)) binds

'nearby' $a_k, a_{k\pm1}$). In particular, it suffices to find all a_k for $k > 0$ and then a_{-k} can be computed from the latter by means of a simple transformation.

7 Essentially periodic and general solutions of overdamped Josephson junction equations

The connection between the functions $E(z), E_{\#}(z), \hat{E}(z)$ pointed out above is important for the lifting the results concerning solutions of Eq. (13) to the level of original OJJE. This procedure applies Eqs. (4), (9), (10), (12) and leads to the following conclusions.

At first, the representation of the two special (and the most important) solutions to OJJE for which the exponents $\exp(\mathrm{i}\varphi)$ are *periodic* (for brevity, we shall call such phase functions *essentially periodic*) follows. It reads

$$\exp(-\mathrm{i}\varphi) = 2\mathrm{i}\omega\left(z\frac{E'(z)}{E(z)} + \frac{n+\epsilon}{2} - \mathrm{i}\varkappa - \mu z,\right), \tag{51}$$

$$\exp(\mathrm{i}\varphi) = -2\mathrm{i}\omega\left(z^{-1}\frac{E'(z^{-1})}{E(z^{-1})} + \frac{n+\epsilon}{2} - \mathrm{i}\varkappa - \mu z^{-1}\right), \tag{52}$$

$$\text{where } z = \exp(\mathrm{i}\omega t).$$

For $\varkappa > 0$ the first of these formulas determines the asymptotic limit (the attractor) of a generic solution whereas the second solution is unstable (the repeller). It is important to emphasize that the functions $\varphi(t)$ defined by Eqs. (51) and (52) are *real* and Eq. (50) is the crucial property utilized in the calculation establishing this fact.

At second, it is straightforward to obtain the 'nonlinear superposition' of solutions (51), (52) operating with their DCHE-related counterparts. The

result is represented by the formula

$$
\begin{aligned}
\exp(\mathrm{i}\varphi) \;=\;& -\frac{\mathrm{i}}{2}\Bigg\{\cos\psi\cdot E(z)+\sin\psi\cdot z^{-\epsilon+2\mathrm{i}\varkappa}\times\\
&\left[E'(z^{-1})+\left(\left(\frac{n+\epsilon}{2}-\mathrm{i}\varkappa\right)z-\mu\right)E(z^{-1})\right]\Bigg\}\times\\
&\Bigg\{\omega\cos\psi\cdot\left[zE'(z)+\left(\frac{n+\epsilon}{2}-\mathrm{i}\varkappa-\mu z\right)E(z)\right]\\
&+\frac{1}{4\omega}\sin\psi\cdot z^{-\epsilon+1+2\mathrm{i}\varkappa}E(z^{-1})\Bigg\}^{-1}, \qquad (53)
\end{aligned}
$$

where ψ is an arbitrary real constant. More exactly, the set of all functions φ described by Eq. (53) is parameterized by a point on the unit circle. As opposed to (51), (52), the function (49) is defined on the universal covering of the Riemann sphere with punctured poles, Ω. Continuous (and then real analytic) function $\varphi(t)$ determined by this equation on $\Omega_1 \in \Omega$, where z is understood as $e^{\mathrm{i}\omega t}$, is just the general solution of OJJE in the case of phase-lock.

In particular, Eqs. (51),(52) arise as particular cases of (53) for $\psi = 0$ and $\psi = \pi/2$, respectively. As a consequence, asymptotic properties of the general solution mentioned above immediately follow. Indeed, as t increases, the exponent (53) is converging to (51) and is moving off (52) (unless it coincides with the latter). The two solutions described by (51),(52) are the only ones which are not affected by the translations $t \to t + 2\pi\omega^{-1}$ (in the sense the exponents (51),(52) are kept unchanged) and preserve their form in asymptotics.

At third, considering φ defined by (51) as analytic function of z and taking in account Eq. (13), one obtains

$$
\begin{aligned}
\frac{\mathrm{d}\varphi}{\mathrm{d}z} \;=\;& -\mathrm{i}z^{-2}\Bigg\{z^3\left(\frac{E'(z)}{E(z)}\right)^2+z\left(\left(1-z^2\right)\mu+z\left((\epsilon-1)-2\mathrm{i}\varkappa\right)\right)\frac{E'(z)}{E(z)}\\
&-\left(z+z\left(\frac{n-\epsilon}{2}+\mathrm{i}\varkappa\right)-\mu\right)\left(\left(\frac{n+\epsilon}{2}-\mathrm{i}\varkappa\right)-\mu z\right)+\frac{z}{4\omega^2}\Bigg\}\times\\
&\left\{z\frac{E'(z)}{E(z)}+\left(\frac{n+\epsilon}{2}-\mathrm{i}\varkappa\right)-\mu z\right\}^{-1} \qquad (54)
\end{aligned}
$$

On the unit circle, this φ verifies OJJE. It is therefore smooth (even real analytic). Then (54) is continuous on the unit circle. Finally, since $\exp \mathrm{i}\varphi|_{z=\exp \mathrm{i}\omega t}$

is periodic, the following proposition holds true:

Proposition 2.

> The quantity
>
> $$k = (2\pi)^{-1} \oint \frac{\mathrm{d}\varphi}{\mathrm{d}z}\, \mathrm{d}\, z, \tag{55}$$
>
> where $\mathrm{d}\varphi/\mathrm{d}\, z$ denotes the RHS expression from Eq. (54) and the integration is carried out over the circle $|z| = 1$, is well defined and equals to an integer.

This integer is the degree of the map $\mathbb{S}^1 \Rightarrow \mathbb{S}^1$ induced by the function (51). In physical applications, it is called *the phase-lock order* and is considered as an integer-valued function of the parameters. Phase-lock order is involved in the formula representing the property of being 'essentially periodic' for the phase function defined by Eq. (51) (and asymptotically for a generic phase function) which reads

$$\forall t\ \varphi(t + 2\pi\omega^{-1}) = \varphi(t) + 2\pi k$$

In a phase-lock state of JJ, the uniformly distributed discrete levels of averaged voltage equal to $k \cdot (\hbar\omega/2e)$ for some $k = 0, \pm1, \pm2 \ldots$ are observed.

Conjecture C.

> Any integer map degree (55) is realized on some non-empty open subset of the space of the problem parameters n, μ, ω, ϵ.

This assertion is closely cognate to the item 1 of the above Conjecture A.

8 Summary

It the present work, the general solution of the overdamped Josephson equation (1) is derived for the (co)sinusoidal RHS function (2) in the case of one of three possible asymptotic behaviors known as the phase-lock mode. The solution is represented in explicit form in terms of the Floquet solution of the particular instance (corresponding to the vanishing of one of the four free constant parameters) of the double confluent Heun equation (DCHE). The Floquet solution of DCHE is represented in terms of the Laurent series whose

coefficients are determined by the convergent infinite products of 2×2 matrices with a single zero element tending to the idempotent matrix (31). The derivation presupposes the existence of a real solution of the transcendental equation (43) which is equivalent to the claim of realization of the phase-lock mode for the given parameter values. The plausible criterion of its existence (i.e. *the phase-lock criterion*) is conjectured.

It is worth summarizing here the main steps of solution of OJJE. They can be condensed as follows.

- The investigation of the basic properties of Eq. (1) for arbitrary periodic (sufficiently regular) $q(t)$ allows one to establish the division of the space of the problem parameters into the two open areas of which one corresponds to the phase-lock property of the OJJE solutions whereas another corresponds to their pseudo-chaotic behavior revealing no stable periodicity. For the (lower-dimensional) complement to these areas the intermediate behavior is observed. The corresponding results are discussed in sufficient details in Ref. [11].

- The next important point is the intimate connection (first mentioned by V. Buchstaber, see, e.g., Ref. [13]) between (1) and a simple *linear* system of the two first order ODEs (5). For (co)sinusoidal RHS function (2), the latter takes the form (3).

- At the next step, the transformation (9) was found which converts the linear system (3) to a particular instance of the double confluent Heun equation (11).

Generally speaking, it could be solved by means of the expansion in Laurent series [14],[12] centered at the singular points but preliminarily the additional simple but important transformation has to be carried out:

- the 'branched' power factor involving unspecified constant ($\varkappa$-dependent contribution in Eq. (12)) is introduced.

- The addition of the discrete 'parity' parameter ϵ, assuming either the value 0 or the value 1, which is involved in the power factor proves necessary for the subsequent exhaustive 'indexing' of the solution space.

- After that, the standard technique of the power expansion leads to the 'endless' sequence of the 3-term constraints (15) (or, which is the same, (17)) imposed on the unknown series coefficients.

- The next step is the devision of the set of power series coefficients into two subsets. The non-negative-index-value coefficients and non-positive-index-value ones are treated separately, solving the separate subsets of the equations (15) and (17), respectively, for $k \geq 1$. The application of the continued fraction technique leads, after some transformations, to the 'explicit' formulas for the series coefficients involving infinite products of 2×2 matrices converging for large index values to the idempotent matrix (31). This convergence is sufficiently fast to imply the convergence of the matrix products and, accordingly, the finiteness of the series coefficients. Moreover, the associated estimates make evident the existence of the absolutely converging majorants for the resulting Laurent series. Therefore, they actually determine the Floquet solution of DCHE. The latter proves representable as the sum of the two entire functions of the arguments z and z^{-1}, respectively.

- The procedure producing Laurent series coefficients noted above proves suffering however from the improper introduction of a kind of vicious singularities arising as zeroes in denominator which appear for some special parameter values. They are eliminated my means of multiplication of the 'raw' coefficient expressions by some z-independent (but parameter dependent) factors given in explicit form.

- Now the 'solution candidate' for Eq. (13) can be represented as the analytic function (38) which is well-defined for any parameter values. However, at the price of automatic convergence of the series it has been built upon from, it does not always verify Eq. (13). The equation is fulfilled if and only if the fulfillment of Eq. (43), which is the transcendental equation for the still unspecified parameter $\varkappa$, is stipulated.

At that stage, having solved Eq. (43), a single solution (the Floquet solution) of DCHE can be regarded as having been explicitly constructed.

- The invariance of the space of solutions of DCHE under consideration with respect to transformation represented by Eq. (45) allows one to immediately obtain the fundamental system of its solutions in terms of the single Floquet solution noted above.

- The automorphism represented by Eq. (48) expresses the important intrinsic property of the Floquet solution of DCHE. It is used for the

derivation of the explicit representation of the exponent $\exp(\mathrm{i}\varphi)$ specifying the real valued phase function φ to be obtained as the restriction of the analytic function from the universal covering of the Riemann sphere with punctured poles (Eq. (53)) to the universal covering of the unit circle. It yields the general solution of Eqs. (1),(2) *in the case of phase-lock*.

- Employing analytic properties of $\exp(\mathrm{i}\varphi)(z)$, the formula (55) involving Floquet solution of DCHE follows which gives the degree of the map $\mathbb{S}^1 \Rightarrow \mathbb{S}^1$ it induces (the winding number) also known in application fields as the phase-lock order.

It is worth noting in conclusion that all the constructions derived above admit a straightforward algorithmic implementation which have been used for the numeric verification of the relevant relationships.

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A Appendix: outline of the Lemma proof

Eq. (26) implies $\beta_j = \mu^2\alpha_{j-1}Z_j^{-1}, \delta_j = \mu^2\gamma_{j-1}Z_j^{-1}$ and hence the asserted properties of the sequences β_j, δ_j follow from the existence of finite limits for the sequences α_j, γ_j. As to α_j and γ_j, they have to obey the identical decoupled 3-term recurrence relations which, for α's, read

$$\alpha_j = (1 + \lambda Z^{-1}_{j+(\epsilon-1)/2})\alpha_{j-1} + \mu^2 Z^{-1}_{j-\tilde{\epsilon}+(\epsilon-1)/2}\alpha_{j-2} \tag{56}$$

where $\tilde{\epsilon} = 1$ for the upper choice of σ in (27) and $\tilde{\epsilon} = 0$ for the lower σ choice therein. It suffices to consider the α-sequence case.

Evidently, for every integer $j_0 > 0$ and $l > 0$ any solution of equations (56) can be represented in terms of the decomposition

$$\alpha_{j_0+l} = (1 + p_{j_0,l})\alpha_{j_0-1} + q_{j_0,l}\alpha_{j_0-2}, \tag{57}$$

for some coefficients $p_{j,l}, q_{j,l}$ independent on the 'starting' terms $\alpha_{j_0-1}, \alpha_{j_0-2}$. Applying (56), it is straightforward to show

$$q_{j_0,l+1} \;=\; (1 + p_{j_0+1,l})\mu^2 Z^{-1}_{j_0-\tilde{\epsilon}}, \tag{58}$$

whereas for $p_{*,*}$ one gets the following recurrence relation:

$$p_{j_0,l+1} = p_{j_0+1,l} + \lambda(1 + p_{j_0+1,l})Z_j^{-1} + \mu^2(1 + p_{j_0+2,l-1})Z^{-1}_{j+1-\tilde{\epsilon}}. \tag{59}$$

Eq. (59) is equivalent to (56), essentially, but it possesses the advantage of being endowed with the standard 'initial conditions'

$$p_{j_0,-1} \;=\; 0, p_{j_0,-2} = -1 \tag{60}$$

which follow from definitions. Besides, one gets

$$q_{j_0,-1} \;=\; 0, q_{j_0,-2} = 1. \tag{61}$$

It proves convenient to carry out one more rearrangement of unknowns introducing the differences

$$\Delta p_{j_0,l} = p_{j_0,l+1} - p_{j_0,l}. \tag{62}$$

which obey the own 'initial conditions'

$$\Delta p_{j_0,-2} \;=\; 1, \tag{63}$$
$$\Delta p_{j_0,-1} \;=\; \lambda Z^{-1}_{j_0}, \tag{64}$$

and similar recurrence relations

$$\Delta p_{j_0,l+1} = \Delta p_{j_0+1,l} + \lambda \Delta p_{j_0+1,l} Z^{-1}_{j_0} + \mu^2 \Delta p_{j_0+2,l-1} Z^{-1}_{j_0+1-\tilde{\epsilon}}. \tag{65}$$

Now, summing up the subset of the latter with the common sum of indices at the left and taking into account (64), all but two 'free' Δp-terms cancel out and one obtains the following equation

$$\begin{aligned}\Delta p_{j_0,l+1} &= \lambda Z^{-1}_{j_0+2+l} \\ &\quad +\lambda \sum_{m=0}^{l+1} \Delta p_{j_0+1+m,l-m} Z^{-1}_{j_0+m} \\ &\quad +\mu^2 \sum_{m=0}^{l+1} \Delta p_{j_0+2+m,l-1-m} Z^{-1}_{j_0+m+1-\tilde{\epsilon}}. \end{aligned} \tag{66}$$

In the sums, the second index of $\Delta p_{*,*}$ is everywhere less than the same index at the left that allows to apply the method of mathematical induction. For the 'starting' values -1,0 of the second index one has

$$\begin{aligned} Z_{j_0}\Delta p_{j_0,-1} &= \lambda, \\ Z_{j_0+1}\Delta p_{j_0,0} &= \lambda(1+\lambda Z^{-1}_{j_0}) + \mu^2 Z_{j_0+1} Z^{-1}_{j_0+2-\tilde{\epsilon}}. \end{aligned}$$

Therefore for $l = -1, 0$ there exist the finite limits $\lim_{j_0\to\infty} |Z_{j_0+l+1}\Delta p_{j_0,l}|$. As a consequence, for these l's one has

$$|\Delta p_{j_0,l}| < \tilde{N} |Z_{j_0+l+1}|^{-1} \tag{67}$$

for appropriate constant $\tilde{N}$ which is convenient to choose > 1. Let us consider this fact as the starting point of mathematical induction and assume that for some integer $l_0 \geq 0$ and any integer l from the interval $[-1, l_0]$, (67) holds true. We may apply it for the estimating from above of the quantity $|\Delta p_{j_0,l_0+1}|$. This can be realized making use of the 'decomposition' (66) and the following elementary inequalities

$$\sum_{m=j_0}^{L+j_0+1} |Z_m|^{-1} < \frac{1+|n+1|^{-1}}{j_0 - |n+1|/2 + (\epsilon-1)/2}, \tag{68}$$

$$\sum_{m=j_0+1-\tilde{\epsilon}}^{L+j_0+2-\tilde{\epsilon}} |Z_m|^{-1} < \frac{1+|n+1|^{-1}}{j_0 - |n+1|/2 + (\epsilon-1)/2}, \tag{69}$$

where $L > 0$ (and $n \neq -1$) . These imply the inequalities

$$\begin{aligned}|\Delta p_{j_0,l_0+1}| &\leq |\lambda||Z_{j_0+l_0+2}|^{-1} \\ &\quad +\tilde{N}|Z_{j_0+l_0+1}|^{-1}\left(|\lambda|\sum_{m=0}^{l_0+1}|Z_{j_0+m}|^{-1}+|\mu^2|\sum_{m=0}^{l_0+1}|Z_{j_0+m+1-\tilde{\epsilon}}|^{-1}\right) \\ &< |Z_{j_0+l_0+2}|^{-1}\left(1+\tilde{N}\frac{|Z_{j_0+l_0+2}|}{|Z_{j_0+l_0+1}|}\frac{(|\lambda|+|\mu^2|)(1+|n+1|^{-1})}{(j_0-|n+1|/2+(\epsilon-1)/2)}\right).\end{aligned}$$

Since we assumed $\tilde{N} > 1$, there exists the lower index value bound such that for any j_0 exceeding it the factor in brackets is less than $\tilde{N}$ and then the above inequalities imply $|\Delta p_{j_0,l_0+1}| < \tilde{N}|Z_{j_0+l_0+2}|^{-1}$. (67) is therefore established for sufficiently large j_0 and arbitrary $l \geq 0$. Increasing $\tilde{N}$ if necessary, (67) proves valid for arbitrary j_0.

In view of this property, one sees that the sum $\sum_{l=0}^{\infty}\Delta p_{j_0,l}$ has the majorant $\sum_l |Z_{j_0+l+1}|^{-1}$ and thus converges itself. The sequence of its partial sums $\sum_{m=0}^{l}\Delta p_{j_0,m} = p_{j_0,l+1} - p_{j_0,-1} = p_{j_0,l+1}$ also converges as $l \to \infty$. Moreover, in view of (60), (62), (68), (67) one has the l-uniform bound

$$|p_{j_0,l+1}| < \tilde{N}\sum_{m=0}^{l}|Z_{j_0+m+2}|^{-1} < \frac{\tilde{N}(1+|n+1|^{-1})}{j_0+1+(\epsilon-1)/2}. \tag{70}$$

It follows from Eqs. (57),(58)

$$\alpha_j = \alpha_{j_0+l} = (1+p_{j_0,l})\alpha_{j_0-1} + (1+p_{j_0+1,l-1})\mu^2 Z^{-1}_{j_0-\tilde{\epsilon}}\alpha_{j_0-2} \tag{71}$$

and the convergence of α-sequence follows from the convergence of $p_{*,l}$ as $l \to \infty$. Then one has

$$\lim \alpha_j - \alpha_{j_0-1} = \lim_l p_{j_0,l}\alpha_{j_0-1} + (1+\lim_l p_{j_0+1,l})\mu^2 Z^{-1}_{j_0-\tilde{\epsilon}}\alpha_{j_0-2}. \tag{72}$$

The factors in front of the first and second terms to the right scales as j_0^{-1} and j_0^{-2}, respectively. We may therefore write down the following inequality

$$|\lim \alpha_j - \alpha_{j_0-1}| = N\max(|\alpha_{j_0-1}|, |\alpha_{j_0-2}|)j_0^{-1},$$

where N may depend on the parameters $n, \lambda, \mu, \varkappa, \epsilon$ but not on the specific specimen of α-sequence. This obviously implies the inequality (28).

It has also to be noted in conclusion that the case $n = -1$ formally falling off the above speculation does not actually correspond to an exceptional situation. Although inequalities (68) formally fail, similar ones differing from (68) in the values of 'constant' (j_0-independent) terms alone can be derived. The further reasoning holds true and leads to the same conclusions.